CARDINALITY OF SETS

Definitions – MTH110 Review

- Let A and B be any sets. A has the same cardinality as B, if and only if there is a bijection from A and B or from B to A.
- A set is called finite if, and only if, it is the empty set or there is a bijection from {1, 2, … , n} to it, where n is a positive integer. In the first case, the number of elements in the set is said to be 0 and in the second case it is said to be n. The number of elements of a finite set A is denoted N(A).
- A set that not finite is called infinite.

Properties

- Addition Rule: If $\{A_1, A_2, \ldots, A_n\}$ is a partition of a finite set A, then
  $N(A) = \sum_{i=1}^{n} N(A_i)$
- Difference Rule: If A is a finite set and $B \subseteq A$, then
  $N(A-B) = N(A)-N(B)$
- Inclusion/Exclusion Rule: If A and B are any finite sets, then
  $N(A\cup B) = N(A) + N(B) - N(A\cap B)$

COUNTING AND FUNCTIONS

Pigeonhole Principle

A function from one finite set to a smaller finite set cannot be one-to-one: There must be at least 2 elements in the domain that have the same image in the co-domain.

Generalized Pigeonhole Principle

For any function f from a finite set X to a finite set Y, and for any positive integer k,

if $N(X) > k.N(Y)$,
then $\exists y \in Y$ s.t. y is the image of at least k+1 distinct elements of X.

MULTIPLICATION RULE

If an operation consists of k steps
and for any i from 1 to k, the i^{th} step can be performed in n_i ways,

then the whole operation can be performed in $\prod_{i=1}^{k} n_i$ ways.

PERMUTATIONS AND COMBINATIONS

Permutations

- For any integer $n \geq 1$,
  the number of permutations of a set with n elements is n!
- An r-permutation of a set of n elements is an ordered selection of r elements taken from the set of n elements. The number of r-permutations of a set of n elements is denoted $P(n,r) = \frac{n!}{(n-r)!}$

Combinations

Let n and r be non-negative integers with $r \leq n$.

- An r-combination of a set of n elements is a subset of r of the n elements. The symbol $\binom{n}{r}$ which is read "n choose r" denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.
- $\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$

Permutations of a set with repeated elements

- If a collection consists of n objects grouped into k categories such that all the objects in the same category are indistinguishable from each other and the i^{th} category has n_i elements. Then the number of distinct permutations of the n objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\dots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_{k3}} = \frac{n!}{n_1!n_2!\dots n_k!}$$

Combinations with repetitions allowed

- An r-combination with repetition allowed, or multiset of size r, chosen from a set $X=\{x_1, \dots, x_n\}$ of n elements is an unordered selection of elements taken from X with repetition allowed. This is denoted by $[x_{i1}, \dots, x_{ir}]$ where each x_{ij} is in X and some of the x_{ij} may equal each other.
- The number of multisets of size r selected from a set of n elements is $\binom{r+n-1}{r}$

PROPERTIES OF COMBINATIONS

- For any positive integer n: $\binom{n}{n}$ =1 and $\binom{n}{1} = \binom{n}{n-1}$ =n

Combination of Complement

- Let n and r be non-negative integers with r≤n. Then $\binom{n}{r} = \binom{n}{n-r}$

Pascal's Formula:

- $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

Binomial Theorem

- Given any real numbers a and b and any non-negative integer n,

  $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$